MATH 1650: SECTION 2.1: GRAPHS OF POLYNOMIAL FUNCTIONS

EXPLORATION: Use a graphing utility to graph the following functions. What patterns evolve?

- $f(x) = x^2$
- $f(x) = x^4$
- $f(x) = x^6$

- $f(x) = -x^2$
- $f(x) = 2x^4$
- $f(x) = -\frac{1}{2}x^6$

- $f(x) = x^3$
- $f(x) = x^5$
- $f(x) = x^7$

- $f(x) = -x^3$
- $f(x) = 2x^5$
- $f(x) = -\frac{1}{2}x^7$

If n is even, and $a > 0$, the graph of $f(x) = ax^n$ looks like:

If n is even, and $a < 0$, the graph of $f(x) = ax^n$ looks like:

If n is odd, and $a > 0$, the graph of $f(x) = ax^n$ looks like:

If n is odd, and $a < 0$, the graph of $f(x) = ax^n$ looks like:

POLYNOMIAL FUNCTION: A **polynomial function** is a function of the form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0,$$

where $a_0, a_1, \dots, a_n$ are real numbers and n is a whole number.

NOTE: For $n > 0$, the subscripts here match the power of x. For example, a_1 is the coefficient of $x = x^1$, a_2 is the coefficient of x^2, etc. Constant, linear, and quadratic functions are all part of the larger 'polynomial' family.

EXAMPLE: For $f(x) = 1 - 2x - 4x^3$:

$a_3 =$ $a_2 =$ $a_1 =$ $a_0 =$

TERMINOLOGY: A polynomial function: $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$, with $a_n \neq 0$ has:

- **degree** n
- **leading term** $a_n x^n$
- **leading coefficient** a_n

EXAMPLE: Find the degree, leading term, and leading coefficient of the following polynomial functions.

- $f(x) = -2x^4 + 5x^3 - 9x^2 + 3$
  - degree:
  - leading term:
  - leading coefficient:
- $g(x) = 3 - 2x - x^3$
  - degree:
  - leading term:
  - leading coefficient:

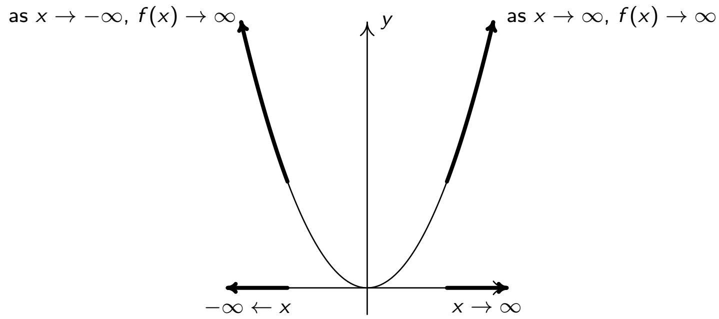
For the next few functions, to find the leading term, multiply out only the leading terms of each factor to get the leading term of the polynomial function. For example to get the leading term of $p(x) = (2x-1)^2(x^3 + x + 1)$, we multiply: $p(x) = (2x)^2(x^3) + \dots = 4x^5 + \dots$ so the leading term is $4x^5$.

- $F(x) = (2x-1)(x^2 + 3x + 1)$
  - degree:
  - leading term:
  - leading coefficient:
- $G(x) = -2(x-1)^3(x+2)^2$
  - degree:
  - leading term:
  - leading coefficient:
- $p(z) = -(3z-1)(z+2)^2\left(z^2+1\right)$
  - degree:
  - leading term:
  - leading coefficient:
- $q(z) = \dfrac{1}{2}(2z+3)^2(z-1)^3$
  - degree:
  - leading term:
  - leading coefficient:

END BEHAVIOR: The **end behavior** of a function $f(x)$ describes the behavior of the $f(x)$ a 'at the ends of the x-axis.' We investigate the end behaviors of $f(x) = x^2$ and $g(x) = x^3$ below.

Looking at the table below on the left, we see that as the inputs, x, grow unbounded in the negative direction, the outputs, $f(x)$, grow unbounded in the positive direction. We describe this by writing: 'as $x \to -\infty$, $f(x) \to \infty$.' Graphically, the farther to the left we travel on the x-axis, the farther up the y-axis the function values travel. This is why we use an 'arrow' on the graph in Quadrant II heading upwards to the left. Similarly, we write 'as $x \to \infty$, $f(x) \to \infty$' since as the x values increase, so do the $f(x)$ values - seemingly without bound. Graphically we indicate this by an arrow on the graph in Quadrant I heading upwards to the right. This behavior holds for all functions $f(x) = x^n$ where $n \geq 2$ is even and corresponds to the '$\cup$' shape observed earlier.

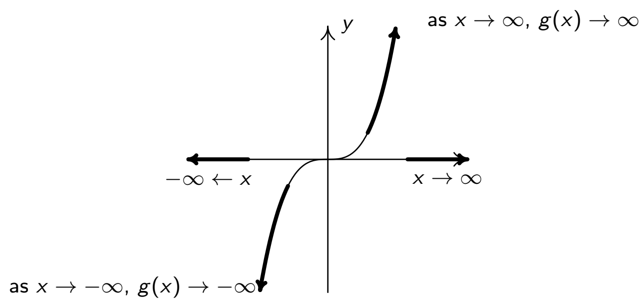
x	$f(x) = x^2$
-1000	1000000
-100	10000
-10	100
0	0
10	100
100	10000
1000	1000000



$f(x) = x^2$

Repeating this investigation for $g(x) = x^3$, we find as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$. This trend holds for all functions $g(x) = x^n$ where n is odd and corresponds to the 's' observed earlier.

x	$g(x) = x^3$
-1000	-1000000000
-100	-1000000
-10	-1000
0	0
10	1000
100	1000000
1000	1000000000



$g(x) = x^3$

EXAMPLE: Use a graphing utility to graph each of the functions below to determine their end behavior.

- $f(x) = x^3 - 4x + 1$
  - as $x \to -\infty$, $f(x) \to$
  - as $x \to \infty$, $f(x) \to$
- $g(t) = -t^4 + 2t^2 - 1$
  - as $t \to -\infty$, $g(t) \to$
  - as $t \to \infty$, $g(t) \to$

Graph each function above along with its leading term on a graphing utility and 'Zoom Out.' What happens?

LEADING TERM TEST: The end behavior of a polynomial matches the end behavior of its leading term.

EXAMPLE: Use the leading term test to predict the end behavior of the following functions. Check your answer using a graphing utility.

- $F(x) = 100x^2 - x^3$
  - leading term:
  - as $x \to -\infty,\ F(x) \to$
  - as $x \to \quad \infty,\ F(x) \to$

- $G(x) = 0.01x^4 - 100x^3 + 1$
  - leading term:
  - as $x \to -\infty,\ G(x) \to$
  - as $x \to \quad \infty,\ G(x) \to$

- $h(t) = (t+1)(2-t)(t+3)$
  - leading term:
  - as $t \to -\infty,\ h(t) \to$
  - as $t \to \quad \infty,\ h(t) \to$

- $p(x) = 3(x-5)^2(x^2+x+1)$
  - leading term:
  - as $x \to -\infty,\ p(x) \to$
  - as $x \to \quad \infty,\ p(x) \to$

- $p(z) = (1-z)^3(2z+1)$
  - leading term:
  - as $z \to -\infty,\ p(z) \to$
  - as $z \to \quad \infty,\ p(z) \to$

- $q(x) = 2(x-1)(3-2x)^2$
  - leading term:
  - as $x \to -\infty,\ q(x) \to$
  - as $x \to \quad \infty,\ q(x) \to$

ZEROS OF POLYNOMIALS:

DEFINITION: The **zeros** of a function f are the solutions to the equation $f(x) = 0$.

NOTE: Geometrically, c is a zero of f means $(c, 0)$ is an x-intercept of the graph of $y = f(x)$.

EXAMPLE: All of the functions below have the same zeros: $x = 0$, $x = 1$, and $x = -2$. For each function, state the multiplicity of each zero and describe what the graph looks like 'near' the corresponding x-intercept. Choose among the adjectives: linear, 'u'-shaped, 's'-shaped. What patterns do you notice?

- $f(x) = x(x-1)(x+2)$

zero	multiplicity	shape
$x = 0$		
$x = 1$		
$x = -2$		

- $f(x) = x^2(x-1)(x+2)$

zero	multiplicity	shape
$x = 0$		
$x = 1$		
$x = -2$		

- $f(x) = x(x-1)^2(x+2)$

zero	multiplicity	shape
$x = 0$		
$x = 1$		
$x = -2$		

- $f(x) = x^2(x-1)(x+2)^2$

zero	multiplicity	shape
$x = 0$		
$x = 1$		
$x = -2$		

- $f(x) = x^3(x-1)(x+2)^2$

zero	multiplicity	shape
$x = 0$		
$x = 1$		
$x = -2$		

- $f(x) = x^4(x-1)(x+2)^3$

zero	multiplicity	shape
$x = 0$		
$x = 1$		
$x = -2$		

If $x = c$ is a zero of f with multiplicity **one**, the graph of $y = f(x)$ is ______________ near $(c, 0)$.

If $x = c$ is a zero of f with **even** multiplicity, the graph of $y = f(x)$ is ______________ near $(c, 0)$.

If $x = c$ is a zero of f with **odd** multiplicity larger than 1, the graph of $y = f(x)$ is ______________ near $(c, 0)$.

EXAMPLE: Putting it all together: let $p(x) = -(x+1)^2(x-2)(x^2+1)$.

- Find the leading term of $p(x)$ and use this to describe the end behavior of the graph of $y = p(x)$.
- Find all real zeros of p and state their multiplicities.
- Find the x-intercepts of the graph of $y = p(x)$ and describe the behavior of the graph near each x-intercept.
- Find $p(0)$ and use this to find the y-intercept of the graph of $y = p(x)$.
- Graph $y = p(x)$ by hand using end behavior, the y-intercept, and behavior near the x-intercepts as a guide.
- Use your hand-drawn graph to make a sign diagram for $p(x)$.
- Check your answer using a graphing utility.

HOMEWORK: Section 2.1: 11 - 30 all